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AN EXACT ESTIMATE FOR THE COEFFICIENTS OF ATTACHED MASSES*

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An inequality strengthening the well-known Polya-Schiffer result /1, 2/ is obtained for an arbitrary system of rigid bodies in translational motion through an infinite potential flow of a perfect incompressible fluid. Non-trivial examples of attaining equality in the estimate are given, establishing an analogy with the inverse problem of the theory of elasticity.

Let us denote by $S^+ = US_i,\ i = 1, 2, \ldots, m$ a three-dimensional region occupied by m perfectly rigid bodies with a Lyapunov boundary $\Gamma = U\Gamma_i$ and total volume $V > 0$. S^- complements S^+ to a complete space and represents a connected region occupied by a fluid of unit density at rest at infinity, **M** is the symmetric tensor of the coefficients of attached masses corresponding to the translational motion of the system of bodies in the direction of the axes of a Cartesian ccordinate system $X_1X_2X_3$, E is a unit tensor, and $I_1(D), I_2(D), I_3(D)$ are the invariants of an arbitrary symmetric tensor, i.e. they are the coefficients of the characteristic polynomial

$$I_1 = d_{11} + d_{22} + d_{33} \tag{1}$$

$$I_2 = d_{12}^2 + d_{23}^2 + d_{13}^2 - d_{11}d_{22} - d_{22}d_{33} - d_{11}d_{33} \tag{2}$$

$$I_3 = d_{11}d_{22}d_{33} + 2d_{12}d_{23}d_{13} - d_{11}d_{23}^2 - d_{22}d_{13}^2 - d_{33}d_{12}^2 \tag{3}$$

The indices 1, 2, 3 will denote indexation over the corresponding axes.

The following theorem was formulated in /1, 2/: under the conditions given above, the mean attached mass of a body or of a system of arbitrary bodies is not less than the corresponding quantity for a sphere of the same volume

$$\frac{1}{3}(\mu_{11}V + \mu_{22}V + \mu_{33}V) \geqslant {}^1/_2 V \tag{4}$$

or, in the equivalent form,

$$I_1(E + \mathbf{M}) \geqslant {}^9/_2 \tag{5}$$

(μ_{lk} are the components of **M**).

The proof of inequality (4) given in /2/ follows from the variational Dirichlet principle

$$K_0 = \sup K(\varphi) \tag{6}$$

$$K(\varphi) = \int\limits_{\Gamma} \varphi u_l n^l\, d\Gamma - \frac{1}{2} \int\limits_{S^+} \frac{\partial\varphi}{\partial x_l} \frac{\partial\varphi}{\partial x^l}\, dx_1\, dx_2\, dx_3$$

where the kinetic energy K_0 is a quadratic form of the components of the velocity vector $\mathbf{u} = (u_1, u_2, u_3)$

$$K_0 = {}^1/_2 V(\mathbf{Mu}, \mathbf{u}) \tag{7}$$

$\varphi(x_1, x_2, x_3)$ is an arbitrary function in S^- decreasing at infinity, and n^l are the components of the vector of the unit normal at any point of Γ directed into the region S^+. In (6) and henceforth, the repeated indices denote summation.

To obtain the estimate (4) we choose the function φ of (6) in the form of a linear combination of the first derivatives with respect to the coordinates of the Newton potential of the masses of unit density distributed within the volume of the bodies

$$\varphi = a^l \frac{\partial\chi}{\partial x_l}, \quad \chi = \int\limits_{S^+} \frac{dx_1'\, dx_2'\, dx_3'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2]^{1/2}}$$

Maximizing $K(\varphi)$ over $a^l\ (l = 1, 2, 3)$ we obtain the inequality

$$K_0 \geqslant \frac{V}{2} b^l u_l^2, \quad b^l = \frac{A_{ll}}{1 - A_{ll}} \tag{8}$$

where A_{ll} are the diagonal elements of the tensor A with components

$$A_{lj} = -\frac{1}{4\pi V} \int\limits_{S^+} \frac{\partial^2\chi}{\partial x_l\, \partial x_j}\, dx_1\, dx_2\, dx_3$$

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From the properties of $\chi(x_1, x_2, x_3)$ it follows that

$$A_{11} + A_{22} + A_{33} = 1 \tag{9}$$

Relation (8) written in the coordinate system whose axes coincide with the principal axes of A becomes an equality, especially in the case of any tri-axial ellipsoid. Further in the proof the estimate (8) becomes coarser, remaining an equality at $n = 1$ only in the case of a sphere.

To sharpen the estimate, we shall write (8), taking (7) into account, in the form

$$((\mathbf{M} - B)\,\mathbf{u}, \mathbf{u}) \geqslant 0 \tag{10}$$

where B is a diagonal matrix with components $b_1, b_2, b_3,$ satisfying the relation

$$b_1 b_2 + b_2 b_3 + b_1 b_3 + 2 b_1 b_2 b_3 = 1 \tag{11}$$

which follows from (8) and (9).

Since the vector $\mathbf{u}$ is arbitrary, we obtain from (10) the conditions of non-negativity of the quadratic form with the matrix $(\mathbf{M} - B)$ /3/

$$\mu_{11} \geqslant b_1,\ \mu_{22} \geqslant b_2,\ \mu_{33} \geqslant b_3 \tag{12}$$

the condition

$$(\mu_{11} - b_1)(\mu_{22} - b_2) - \mu_{12}^2 \geqslant 0 \tag{13}$$

and another two analogous conditions obtained from (13) by interchanging the indices, and finally

$$\det(\mathbf{M} - B) \geqslant 0 \tag{14}$$

From (13) and (12) it follows that

$$\mu_{11}\mu_{22} - \mu_{12}^2 \geqslant b_1 b_2 \tag{15}$$

$$- b_1\mu_{22} - b_2\mu_{11} \geqslant \mu_{12} - \mu_{11}\mu_{22} - b_1 b_2 \tag{16}$$

Combining inequality (15) with other analogous inequalities term by term we obtain, from (2),

$$- I_2(\mathbf{M}) \geqslant b_1 b_2 + b_2 b_3 + b_1 b_3 \tag{17}$$

Writing now (14) in full and taking into account (3), we obtain

$$I_3(\mathbf{M}) \geqslant b_1 b_2 b_3 - b_1 b_2 \mu_{33} - b_2 b_3 \mu_{11} - b_1 b_3 \mu_{22} + b_1(\mu_{22}\mu_{33} - \mu_{23}^2) - b_2(\mu_{11}\mu_{33} - \mu_{13}^2) + b_3(\mu_{11}\mu_{22} - \mu_{12}^2)$$

Using three inequalities of the form (16) and grouping the terms on the right-hand side of the resulting relation, we obtain

$$2I_3(\mathbf{M}) \geqslant b_1(\mu_{22}\mu_{33} - \mu_{23}^2) + b_2(\mu_{11}\mu_{33} - \mu_{13}^2) + b_3(\mu_{11}\mu_{22} - \mu_{12}^2) - b_1 b_2 b_3$$

From (15) it now follows that

$$2I_3(\mathbf{M}) \geqslant 2 b_1 b_2 b_3 \tag{18}$$

Combining (17) and (18) we obtain, taking into account (11), the invariant estimate for the coefficients μ_{ik}

$$2I_3(\mathbf{M}) - I_2(\mathbf{M}) \geqslant 1 \tag{19}$$

It can be directly confirmed that the following identity holds for any non-degenerate third-order matrix:

$$I_1((E + D)^{-1}) = \frac{3 + 2I_1(D) + I_2(D)}{1 + I_1(D) + I_2(D) + I_3(D)}$$

which can be used to reduce (19) to its equivalent form

$$I_1((E + \mathbf{M})^{-1}) \leqslant 2 \tag{20}$$

linear with respect to the elements $(E + \mathbf{M})^{-1}$. It is evident that the inequalities (19), (20) hold for any triaxial ellipsoid.

Having written the relation connecting the harmonic and arithmetic means /4/ of the components of the tensor $(E + M)$ in its diagonal form

$$\frac{3 + \mu_{11} + \mu_{22} + \mu_{33}}{3} = \frac{I_1(E + \mathbf{M})}{3} \geqslant 3\left(\frac{1}{1 + \mu_{11}} + \frac{1}{1 + \mu_{22}} + \frac{1}{1 + \mu_{33}}\right)^{-1} = \frac{3}{I_1((E + \mathbf{M})^{-1})}$$

we obtain, from (20), the Polya-Schiffer inequality. The converse is clearly false.

For a two-dimensional tensor of the coefficients Λ the estimate of the type (20) has the form $I_1((E + \Lambda)^{-1}) \leqslant 1$ or, which amounts fo the same thing, $I_2(\Lambda) \geqslant 1$. As was shown in /5/, the equality sign is obtained at the stationary point (for the small variations in the form of the boundary) of the functional of attached mass in the x_1 direction, provided that the total area of the system of bodies and their attached mass in the x_2 direction are given.

The proof of condition (20) shows that the inequality becomes an equality on bodies of such form, that $\partial^2\chi/\partial x_i\partial x_j$ are constant in S^+, or by virtue of the continuous character of χ and its first derivatives, when the following relations hold on the boundary:

$$\varphi(x_1, x_2, x_3) = a^l x_l^2 + l_i;\ x = (x_1, x_2, x_3) \in \Gamma_i$$
$$\partial\varphi/\partial n = 2a^l x_l n^l,\quad n = (n^1, n^2, n^3),\quad i = 1,\ 2, \ldots, m$$

which are also generated by the inverse problem of the theory of elasticity dealing with the optimization of the state of stress of the homogeneous, isotropic, linearly elastic space S^- with cavities loaded at infinity along the axes by the forces q_l $(l = 1, 2, 3)$. By the optimization we mean the control of the form of the boundary resulting in attainment of the least possible local Mises criterion, i.e. the maximum of the second invariant deviator of the stress tensor in S^-. The functions $\partial\chi/\partial x$ have the meaning of elastic displacements of the points of the medium along the axes $2a_l = (Q - 2q_l)/q_l$, $Q = q_1 + q_2 + q_3$, and the constants C_l remain undetermined. For such a boundary M and A are reduced simultaneously to diagonal form.

Unlike the plane case /5/, the actual determination of the boundary at $m > 1$ is very complicated. In the axisymmetric variant $(q_1 = q_2, \mu_{11} = \mu_{22})$ we propose in /7/ a non-linear integral equation in coordinates of the points lying on the meridian section of the boundary, as functions of the arc lengths, and gives the results of its numerical solution.

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ON CERTAIN FEATURES OF THE FLOWS OF VISCOUS COMPRESSIBLE FLUIDS IN CYLINDRICAL PIPES*

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The flow of a viscous compressible fluid in cylindrical pipes when there is volume viscosity /1/ is studied. The process is assumed to be barotropic, as is the case when, for example, heat emission can be neglected or when the fluid has high thermal conductivity. The problem of the correct boundary conditions for the system of defining equations is discussed. The problem of the flow of fluid with Tate's equation of state is solved using the method of separation of variables. Proofs of the existence and uniqueness of the solutions of the ordinary differential equations obtianed are given. The asymptotic behaviour of the velocity as the volume viscosity increased is studied. The coefficients of the volume and shear viscosity are assumed to be constant everywhere.

1. We shall consider, side by side, the plane and the three-dimensional problem of a one-dimensional steady flow in a cylindrical region enclosed between fixed walls. The defining system of equations (Navier-Stokes, continuity and state) is reduced to

$$\rho u u_{,x} = [-p + \zeta u_{,x}]_{,x} + \eta_s(\partial_x^2 + \Delta_j)u \tag{1.1}$$
$$0 = \mathrm{grad}_j[-p + \zeta u_{,x}]$$
$$(\rho u)_{,x} = 0,\ p = p(\rho);\ \zeta = \eta_v + {}^1/_3\eta_s$$

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